Generalized Unit Value Indices

by

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Abstract: The overall price change of an economy is usually measured by averaging the individual items' price changes. As an alternative, one could measure the overall price change by computing average price levels for each period and then comparing these price levels. This change-in-price-levels approach has received scant attention in past research.

The present study challenges this tradition. It develops the family of generalized unit value indices and demonstrates that this family represents a collection of well-known, hardly known, and hitherto unknown price index formulae. All members of this family can be viewed as representatives of the change-in-price-levels approach. This study also examines their axiomatic record and discusses the implications for official price measurement.

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1 Introduction

National statistical institutes typically measure the price change between some comparison and base period in a two stage process. The process starts on the *elementary level* where the price changes of individual items are computed. An item is a class of products that all serve the same purpose, e.g. chocolate bars (dairy milk; weight 200g). An item's price change is measured as the item's average price level of the comparison period divided by the item's average price level of the base period. The concept of comparing price levels is denoted here as the *CPL-approach* (short for "Change in Price Levels"). The average price levels are usually computed as unit values and their ratio is known as the unit value index. An item's unit value is obtained from dividing the item's total expenditures observed during a period by the item's total quantity sold during that period.

On the *upper levels* of price measurement the overall price change of the economy is computed. For this purpose, the unit values calculated at the elementary level are used as the items' prices. Most national statistical institutes use for the upper level computations a Laspeyres-type index. In the literature, many other price indices have been proposed and a few of them have been adopted (e.g., Statistics Sweden computes a Walsh index). Even though the views on the best price index vary, there is widespread agreement that a sensible price index should be such that it can be formulated as a weighted average of the individual items' price changes. This approach to price measurement is denoted here as the *APC-approach* (short for "Average of Price Changes"). Both the Laspeyres and the Paasche index are consistent with this approach.

National statistical offices attempt to design their price measurement process as one integrated process. However, it is a notorious deficiency of existing price measurement procedures that on the elementary level they use the CPL-approach (unit value index), but on the upper levels they usually switch to the APC-approach (e.g., Laspeyres-type index). A very practical problem, having its origin in this conceptual break, is the computation of the overall price change of items that serve the same purpose. This is the borderline between the elementary level and the upper levels of price measurement. Should such a computation follow the CPL-approach or the APC-approach?

This question has been addressed in several studies, including de Haan (2001, 2004), Dalén (2001), and Silver (2009). They suggest applying an expanded form of the unit value index which they denote as "quality-adjusted unit value index". The expansion comes in the form of "quality adjustment factors" that take care of the quality differentials between the items. According to these studies, the estimation of the quality adjustment factors ideally would be based on hedonic regression techniques. These necessitate external information such as the qualitative characteristics of the items. In the absence of such information, de Haan (2001, p. 24) suggests the consideration of a period in which the items are available on the market and are preferably in equilibrium. Then, the price differentials of the items can be used as an estimator for their quality differentials.

This suggestion is the point of departure for the present study. It contributes to the price measurement literature in several ways.

1) It elaborates the proposal of de Haan (2001, p. 24) in a systematic way. It demonstrates that de Haan's proposal is in tune with earlier arguments of Lehr (1885, pp. 37-39) and Davies (1924, pp. 182-186) that lead them to the derivation of their price index formulae, the Lehr index and the Davies index. The present study develops additional price index formulae consistent with the basic ideas of Lehr and Davies and raises our awareness of their inspiring work.

2) In addition, it bundles the old and new price index formulae into a unifying concept: the family of *generalized unit value indices*. The members of this family differ with respect to the precise manner in which their *transformation factors* (a more general label for "quality adjustment factors") are computed. However, they all share the property that the calculation of the transformation factors requires no external information (e.g., qualitative characteristics). Their calculation is exclusively based upon the ordinary price and quantity information of the observed items.

3) The family of generalized unit value indices is an extremely useful tool for aggregating the prices of similar items when no external information on their qualitative differentials is available. However, Lehr (1885, pp. 37-38) and Davies (1924, pp. 182-183) argue that their respective index formula (both are members of the family of generalized unit value indices) can be equally applied to the aggregation of heterogeneous items. This study extends this claim to the complete family of generalized unit value indices. It demonstrates that after minor re-arrangements some members of the family of generalized unit value indices simplify to very traditional price indices, among them the Laspeyres index and the Paasche index. This reinforces the claim that also in the context of heterogeneous items the family of generalized unit value indices generates meaningful results.

4) The generalized unit value indices follow the CPL-approach. Therefore, this study can be viewed as a rehabilitation of the CPL-approach. In order to lend additional support

to this rehabilitation, this study provides a thorough axiomatic analysis of all generalized unit value indices. It demonstrates that the members of this index family have a solid axiomatic record.

5) This study has reassuring news for national statistical institutes and their efforts to use a coherent measurement process. The elementary level of official price measurement usually applies the unit value index. This index follows the CPL-approach. The upper levels of official price measurement often utilize the Laspeyres and the Paasche index, the two most prominent representatives of the APC-approach. As pointed out above, this study demonstrates that both index formulae are members of the family of generalized unit value indices. Therefore, they are consistent with the CPL-approach too. This implies that official price measurement is more coherent than suspected. On all stages, it applies price indices consistent with the CPL-approach.

The paper proceeds as follows. Section 2 provides a brief and critical discussion of the price statistical objections raised against the notion of price levels, and therefore, against the CPL-approach. Beginning with Section 3, the paper proceeds to develop price index formulae that are based on the change of price levels (CPL-approach). The starting point is the unit value index. However, for measuring the change in a price level of a set of items that are not perfectly homogeneous, this simple index formula cannot be directly applied. Therefore, in Section 3 the concept of the *amended unit value index* is introduced. This type of price index can be employed in the context of homogeneous or almost homogeneous items, though not in the context of strongly heterogeneous items. The latter requires an additional refinement. This refinement leads to the family of generalized unit value indices. The basic idea and the definition of this index family are presented in Section 4 along with a summary of the existing literature related to this issue. Section 5 provides a more detailed discussion of this family of price indices. A comparison to some of the traditional index formulae (e.g., Fisher index) can be found in Section 6. The axiomatic properties of the generalized unit value indices are explored in Section 7 and are compared to those of the most popular traditional price indices. Section 8 provides some remarks on promising areas of future research.

2 Fisher's Reservations against the CPL-Approach

The purpose of a price index is to measure the overall price change between a base period $t = 0$ and a comparison period $t = 1$. Let p_i^t denote the price of item i observed in period t. Similarly, let x_i^t denote the quantity of this item purchased in period t. It is assumed that in both periods the same N items are sold in the marketplace. Following the APC-approach, one measures the overall price change by first computing the N items' individual price changes and then averaging these price changes to obtain some overall price change. As an alternative to the APC-approach, one could compute the price levels of the base period and the comparison period and from the ratio of these two price levels the overall price change. This is the CPL-approach and it necessitates the computation of separate price levels for the two periods to be compared. However, in his seminal book on price statistics, Fisher (1922) firmly rejects the notion of a price level. He acknowledges that it can be computed but that it "*... is apt, in general, to prove a delusion and a snare. The reason is that an average of prices of wheat, coal, cloth, lumber, etc. is an average of incommensurables and therefore has no fixed numerical value* (p. 451)."

How the prices of incommensurables can be aggregated into a meaningful average price level is described in Section 4. The approach taken in this study generalizes the method outlined in Lehr (1885, pp. 37-39) and Davies (1924, pp. 182-186). Of course, Fisher is correct in saying that one cannot assign a meaningful numerical value to a price level when this price level is looked at in isolation. However, this is no justification to discard the notion of a price level. The issue is a familiar one from microeconomic price theory. An item's price looked at in isolation is meaningless. However, defining one item as the numeraire good and measuring the prices of all other items in terms of this numeraire good, gives all these prices a meaningful interpretation. Similarly, one could define some period as the "numeraire period". Usually, this period is the base period. The value of the price level of some comparison period can be interpreted relative to the price level of the base period. Defining a suitable price index formula along these lines, one obtains a measure for the percentage change in the price level between the base period and the comparison period. Such a measure is far from meaningless and it is based on the CPL-approach instead of the APC-approach.

Half a century after Fisher had published his reservations against price level measurement, his view was supported by a formal argument from axiomatic index theory. This formal argument can be found in Eichhorn and Voeller (1976, pp. 75-78). Very similar axiomatic objections are presented in Eichhorn (1978, pp. 144-146), Diewert (1993, pp. 7-9), and ILO et al. (2004, p. 292). All of these studies argue that the CPL-approach is based on the concept of unilateral price indices. This is a formula that computes a period's price level solely on the basis of the prices and quantities of that period. Next, all these studies argue that the concept of a unilateral price index is flawed, because no unilateral price index can exist that satisfies a set of indispensable axioms that includes the commensurability axiom. This axiom postulates that the index should be invariant with respect to the units in which the quantities are measured.

There are two problems with this line of reasoning. First, the refutation of the concept of unilateral price indices is not convincing. Auer (2009a) demonstrates that the commensurability axiom, even though indispensable in the context of bilateral price indices (e.g., Fisher index, family of generalized unit value indices), is misplaced in the context of unilateral price indices. Unilateral price indices satisfying the commensurability axiom should be avoided. Second, even if one accepted the refutation of unilateral price indices, this could not serve as an objection against the CPL-approach. Price indices associated with the CPL-approach are usually not ratios of two unilateral price indices, the family of generalized unit value indices being a point in case (see Section 4).

Therefore, this study takes a fresh look at developing price index formulae that follow the CPL-approach. In this context, it develops the family of generalized unit value indices and it demonstrates that the Laspeyres and the Paasche index are members of this index family. Another member of this family is the Banerjee (1977, p. 27) index:

$$\mathrm{P_B} = \frac{\mathrm{V}^1}{\mathrm{V}^0} \frac{(\mathrm{V}^0 + \mathrm{V}^{10})}{(\mathrm{V}^1 + \mathrm{V}^{01})} , \tag{1}$$

where $\mathrm{V}^{\mathrm{t}} = \sum \mathrm{p}_{\mathrm{i}}^{\mathrm{t}} \mathrm{x}_{\mathrm{i}}^{\mathrm{t}}$ and $\mathrm{V}^{\mathrm{st}} = \sum \mathrm{p}_{\mathrm{i}}^{\mathrm{s}} \mathrm{x}_{\mathrm{i}}^{\mathrm{t}}$. This index is almost identical to the Fisher index, since the latter can be expressed in the form

$$\mathrm{P_F} = \frac{\mathrm{V}^1}{\mathrm{V}^0} \frac{\sqrt{\mathrm{V}^0 \mathrm{V}^{10}}}{\sqrt{\mathrm{V}^1 \mathrm{V}^{01}}} . \tag{2}$$

The only difference between the Fisher index (2) and the Banerjee index (1) is the method of averaging the values V^{t} and V^{st}. Where the Fisher index uses a geometric mean, the Banerjee index uses an arithmetic mean. Since the Fisher index is often praised as the best existing price index formula, the Banerjee index (1) and the other members of the family of generalized unit value indices deserve unprejudiced consideration.

3 Amended Unit Value Index

As pointed out above, the current literature favors price indices consistent with the APC-approach. Many of the most popular price indices (e.g., Walsh index) are exclusively embedded in the APC-approach. Some price indices exist that are in line with both the APC- and the CPL-approach. As will be demonstrated in Section 5, the Laspeyres and the Paasche index belong to this class of price indices. There are some price indices which can neither be associated with the APC-approach nor with the CPL-approach (e.g., Stuvel index).

It is useful to begin the analysis with a particularly simple case of price measurement: All price and quantity observations of the base and comparison period refer to the prices and quantities of homogeneous items. Fisher (1923, p. 743) acknowledges that for this case not only the APC- but also the CPL-approach could be followed. Official price statistics, as documented in ILO et al. (2004, p. 164), explicitly recommend the CPL-approach as the best method to aggregate the prices of homogeneous items. More specifically, it is suggested to compute for each period separate unit values P_{UV}^{t} as defined by Segnitz (1870, p. 184):

$$\begin{aligned} P_{UV}^{t} &= \left(\sum p_i^t x_i^t\right) / \left(\sum x_i^t\right) \\ &= V^t / \left(\sum x_i^t\right) \\ &= \sum \left[x_i^t / \left(\sum x_j^t\right)\right] p_i^t \ . \end{aligned} \tag{3}$$

To calculate the overall price change between the base period $t = 0$ and the comparison period $t = 1$ one can use Drobisch's (1871a, p. 39; 1871b, p. 149) unit value index. Using Equation (3), this price index can be expressed as

$$P_{UV} = \frac{P_{UV}^{1}}{P_{UV}^{0}} = \frac{V^1}{V^0} \frac{\sum x_i^0}{\sum x_i^1} \, . \tag{4}$$

An axiomatic justification for the use of the unit value index can be found in Auer (2009b).

Price index (4) was recommended for the case in which the overall price change is calculated from price observations that all relate to homogeneous items. For this case the unit value index is usually regarded as the preferable solution. The unit value index and, as a consequence, also the CPL-approach also lends itself for the case of "almost homogeneous items", that is, for items that differ only with respect to the location of purchase and the moment of purchase within period t. Of course, an amendment of the basic unit value index becomes necessary, when the items exhibit more significant heterogeneity. For example, the items could differ with respect to the size of their packages. In order to make the observations comparable, the unit value index must be amended by transformation factors $\mathrm{z_i}$. The value of each item's transformation factor $\mathrm{z_i}$ depends on the item's package size.

An example may illustrate the idea and the mechanics of the amended unit value index. Suppose that two items are considered. The first is a chocolate bar of 200g and the other is a chocolate bar of 300g. It is assumed that apart from their weights no differences exist between the two items. Producers are indifferent between producing two bars of 300g and three bars of 200g. Also consumers are indifferent between consuming two bars of 300g and three bars of 200g, but not all consumers have both types of chocolate bars available for purchase. The transformation factors are denoted by $\mathrm{z_{big}}$ and $\mathrm{z_{small}}$. The observed prices and quantities are stated in Table 1.

Table 1: Numerical Example for the Mechanics of the Amended Unit Value Index.

	base period		comparison period	
	Price	quantity	price	quantity
big chocolate bar (300g)	8	2	8	3
small chocolate bar (200g)	4	2	6	2

One can choose either an arbitrary number for the transformation factor $\mathrm{z_{big}}$ or for the transformation factor $\mathrm{z_{small}}$. The relationship between $\mathrm{z_{big}}$ and $\mathrm{z_{small}}$, however, is determined by the weight ratio of the two types of chocolate bars:

$$\frac{\mathrm{z_{big}}}{\mathrm{z_{small}}} = \frac{3}{2} \,. \tag{5}$$

One could interpret the ratio of transformation factors as the items' exchange ratio, since two units of 300g-bars are equivalent to three units of 200g-bars:

$$2 \cdot \mathrm{z_{big}} = 3 \cdot \mathrm{z_{small}} \,.$$

In the following, the ratios of transformation factors are denoted as *ratios of equivalence*.

How are the transformation factors incorporated into the unit value formula? The appropriate definition is

$$P_{AUV}^{t} = \left[\sum (p_i^t/z_i)\, x_i^t z_i\right] / \left(\sum x_i^t\, z_i\right) \tag{6}$$

$$= V^t / \left(\sum x_i^t z_i\right). \tag{7}$$

In the chocolate example, choosing a number for z_{big} automatically determines the corresponding number for z_{small}. For $z_{big} = 3$, and therefore $z_{small} = 2$, formulae (6) and (7) would translate all original price and quantity data (p_i^t and x_i^t) into prices and quantities related to 100g-units of chocolate. The transformed quantity $x_{small}^0 z_{small} = 2 \cdot 2 = 4$ is the number of 100g-units sold during the base period in the form of 200g-bars. Correspondingly, the transformed price $p_{small}^0 / z_{small} = 4/2 = 2$ is the base period price of a 100g-unit purchased in the form of a 200g-bar. In the comparison period, this price would be $p_{small}^1 / z_{small} = 6/2 = 3$. Using the transformation factors, also the amended unit values P_{AUV}^0 and P_{AUV}^1 relate to a 100g-unit. Formula (7) produces the values

$$P_{AUV}^1 = \frac{\sum p_i^1 x_i^1}{\sum x_i^1 z_i} = 36/(2 \cdot 2 + 3 \cdot 3) = 2.769$$

$$P_{AUV}^0 = \frac{\sum p_i^0 x_i^0}{\sum x_i^0 z_i} = 24/(2 \cdot 2 + 2 \cdot 3) = 2.4\,.$$

The amended unit value index (AUV) is defined as

$$P_{AUV} = \frac{P_{AUV}^1}{P_{AUV}^0}\,. \tag{8}$$

Using formula (7), the amended unit value index (8) can be expressed in the form

$$P_{AUV} = \frac{V^1}{V^0} \frac{\sum x_i^0 z_i}{\sum x_i^1\, z_i}\,. \tag{9}$$

With identical package sizes, all transformation factors would have the same value: $z_i = z$. In this case the amended unit value index (9) would simplify to the unit value index (4). Formula (9) reveals that the transformation factors affect only the second ratio on the right hand side and that these factors are irrelevant for cases in which the relative quantities remain constant over time.

As previously stated, one transformation factor can be arbitrarily chosen. As long as the correct ratio of transformation factors is used, formula (8) applied to the chocolate example always generates the value

$$\mathrm{P_{AUV}} = \frac{\mathrm{P^1_{AUV}}}{\mathrm{P^0_{AUV}}} = 1.154\ .$$

This number indicates that the overall price level of chocolate bars has increased by 15.4 per cent.

As an alternative one could have used the transformation factors $\mathrm{z_{big}}$ = 3/10 and $\mathrm{z_{small}}$ = 2/10. These factors translate all prices and quantities into 1000g-units of chocolate. This is also the total weight of the base period's basket of items. Another admissible variant one obtains when the purchasing power of a unit of money during the base period is used as a reference. The total monetary value of the base period basket is $\mathrm{V^0}$= 24. Since the total weight of the basket is 1000g, one unit of money buys 1000g/24 ≈ 41.66g. For

$$\mathrm{z_{big}} = 24(3/10) = 7.2$$

prices and quantities relate to 41.66g-units. Correspondingly, Equation (5) yields

$$\mathrm{z_{small}} = (2/3)24(3/10) = 4.8\ .$$

The transformed price $\mathrm{p^0_{small}/z_{small}}$ = 4/4.8 = 5/6 is the base period price of a 41.66g-unit of chocolate purchased in the form of a 200g-bar. Again the amended unit value index generates the value $\mathrm{P_{AUV}}$ = 1.154.

4 Generalized Unit Value Index

The ratios of equivalence (i.e., the ratio of transformation factors) affect the value of the amended unit value index $\mathrm{P_{AUV}}$. In the chocolate example, the ratios were directly determined by the weight ratio of the two items. How could one proceed, however, when the weight ratio of the small and the large chocolate bars were unknown or/and the two types of chocolate bars were of different taste and quality?

The amended unit value index with its transformation factors serves as a natural starting point. Having no information about the appropriate values of $\mathrm{z_i}$, one could try to obtain reasonable estimates $\mathrm{\hat{z}_i}$. Several possibilities exist for obtaining such estimates. These possibilities define a new class of price indices. Accordingly, this class is denoted as the family of generalized unit value indices (GUV):

$$\mathrm{P_{GUV}} = \frac{\mathrm{P^1_{GUV}}}{\mathrm{P^0_{GUV}}} = \frac{[\sum(\mathrm{p^1_i}/\mathrm{\hat{z}_i})\mathrm{x^1_i}\mathrm{\hat{z}_i}]/(\sum \mathrm{x^1_i}\mathrm{\hat{z}_i})}{[\sum(\mathrm{p^0_i}/\mathrm{\hat{z}_i})\mathrm{x^0_i}\mathrm{\hat{z}_i}]/(\sum \mathrm{x^0_i}\mathrm{\hat{z}_i})} = \frac{\mathrm{V^1}}{\mathrm{V^0}}\frac{\sum \mathrm{x^0_i}\mathrm{\hat{z}_i}}{\sum \mathrm{x^1_i}\mathrm{\hat{z}_i}}\ . \tag{10}$$

The formal structure of this class of price indices is identical to the amended unit value index P_{AUV} defined in Equation (8) and expressed in even simpler form in Equation (9). This is the reason why the amended unit value index was introduced in the first place. So far, the only difference is the "hat" appearing on all transformation factors. The hat emphasizes that in the generalized unit value index the values of the transformation factors are not given by known weight ratios or other external information but are estimated.

It is a particularly attractive feature of this index family that the decomposition of the value ratio in a price index and a quantity index is directly given by Formula (10):

$$\frac{V^1}{V^0} = P_{GUV} \frac{\sum x_i^1 \hat{z}_i}{\sum x_i^0 \hat{z}_i} .$$

The quantity component is simply the ratio of the transformed quantities.

Formula (10) is not novel. It goes back to Dalén (2001, p. 11) who generalizes a proposal of de Haan (2001, p. 24). Further elaborations together with empirical applications can be found in de Haan (2004, p. 7) and Silver (2009, p. 11). In all of these studies, the formula is motivated by the problem of aggregating the price movements of similar items (products serving the same purpose but being of different quality) into some average price change. Accordingly, the formula is labeled as "quality-adjusted unit value index". The studies point out that the estimation of the $\hat{z}_i$-values ideally would be based on hedonic regression techniques. These necessitate external information such as the qualitative characteristics of the items. But how can one proceed when, as assumed in the present study, besides prices p_i^t and quantities x_i^t no additional information is available?

For such a situation de Haan (2001, p. 24) commends the consideration of a period in which the relevant items are available on the market and are preferably in equilibrium. The respective price ratio of the items can be used as an estimator for the ratio of the items' transformation factors. The present study systematically elaborates this proposal.

Furthermore, this study makes and justifies a seemingly bold claim: The proposal of de Haan can be equally applied to the case of heterogeneous items. The estimated transformation factors turn the original quantity units into comparable quantity units and the resulting quantities can be meaningfully summed. Dividing total expenditures of a period by this sum yields the average price level of that period. In Section 5 it is demonstrated that this approach leads to price index formulae that can be compared to the traditional price index formulae (e.g., Fisher, Laspeyres, and Paasche index).

Of course, the problem with summing the quantities of heterogeneous items is the incommensurable nature of the quantity units. In the context of similar items, the quantity units can be transformed into quality-adjusted units and these can be readily added. Even simpler is the case of homogeneous or almost homogeneous items. The quantity units can be directly added. Therefore, the key question is Which transformation factors allow for the summation of the quantities of heterogeneous items?

Intuitively, one would request that the quantity units must be physically identical (e.g., 100g of the same chocolate). With homogeneous items this condition is automatically satisfied. In the context of similar items, the use of quality adjustment factors is an attempt to mimic a situation with physically identical units. However, are physically identical units really necessary for a meaningful summation? Physically identical units are attractive, because they ensure that the consumers attach equal valuation to these units and this equivalence in valuation allows for a meaningful summation. This implies, however, that physical identity is not necessary, equivalence in valuation is sufficient.

This insight is not new. It can be found in an early contribution by Lehr (1885, pp. 37-39) that was written in German. Not being aware of that publication, Davies (1924, pp. 183-185) re-invented Lehr's approach. Unfortunately, both studies did not receive the attention they deserved. Possibly, their approach was considered as being too unorthodox. However, it would be a mistake to dismiss the approach prematurely. Whether the approach is sensible or not should be judged on the basis of its results, that is, on the basis of the price index formulae that one obtains when the approach is put into action. If these price index formulae are sensible, then this also justifies the underlying approach. The present paper demonstrates that the approach leads to such sensible price index formulae.

5 Family Members of the Generalized Unit Value Index

In the chocolate example, the base period price of the large chocolate bar is twice as large as the base period price of the small chocolate bar (the price ratio is 8:4). Therefore, one could suspect that the items' ratio of equivalence is also 8:4 = 2. Following this line of reasoning, the ratio of the transformation factors would be

$$\frac{\hat{z}_{\mathrm{big}}}{\hat{z}_{\mathrm{small}}} = \frac{p_{\mathrm{big}}^{0}}{p_{\mathrm{small}}^{0}} = 2\,. \tag{11}$$

Accordingly, one could replace in Equation (10) all transformation factors by the respective base period prices: $\hat{z}_i = p_i^0$. Interestingly, the resulting generalized unit value index is the Paasche index (P_P):

$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum p_i^0 x_i^0}{\sum p_i^0 x_i^1} = \frac{\sum p_i^1 x_i^1}{\sum p_i^0 x_i^1} = P_P \,. \tag{12}$$

In other words, the Paasche index is a generalized unit value index with transformation factors estimated from the prices of the base period only. Therefore, the Paasche index is consistent with the CPL-approach.

In Equation (11), the ratios of equivalence have been derived from the price ratios of the base period. However, when the items are available in both the base and the comparison period (as is usually assumed in index theory), then, instead of looking at the base period prices, one could also look at the prices prevailing during the comparison period. In the chocolate example, the price ratio is 8:6. The corresponding ratio of equivalence would be

$$\frac{\hat{z}_{big}}{\hat{z}_{small}} = \frac{p_{big}^1}{p_{small}^1} = \frac{4}{3} \,. \tag{13}$$

Accordingly, one could replace in Equation (10) all transformation factors by the respective comparison period prices: $\hat{z}_i = p_i^1$. The resulting generalized unit value index is the Laspeyres index (P_L):

$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum p_i^1 x_i^0}{\sum p_i^1 x_i^1} = \frac{\sum p_i^1 x_i^0}{\sum p_i^0 x_i^0} = P_L \,. \tag{14}$$

This demonstrates that also the Laspeyres index is consistent with the CPL-approach. It can be interpreted as the generalized unit value index which estimates its transformation factors exclusively from the prices of the comparison period.

The generalized unit value indices (12) and (14) produce different results. One approach to deal with such ambiguities is to average the two index formulae. Taking the arithmetic mean of estimates (12) and (14) yields the Drobisch index,

$$P_D = (P_L + P_P)/2 \,,$$

and taking the geometric mean yields the Fisher index,

$$P_F = \sqrt{P_L P_P} \,.$$

The problem of ambiguity is well familiar from the APC-approach. There, the ambiguity exists with respect to the appropriate weights to be used for averaging the individual price ratios. In response to this problem, the weights are usually computed from the data of both periods and not just from one period. The same principle can be used in the context of the CPL-approach.

The calculation of the generalized unit value index (10) requires a single number for each ratio of equivalence (i.e., each ratio of transformation factors). If the price ratio of the chocolate bars were constant over time, the values generated by (11) and (13) would both produce the same number. However, in the example considered, there is no such proportionality. One way to obtain a single number for each ratio of equivalence is to average the ratios (11) and (13) in one way or the other. Taking the arithmetic mean (known as the Carli index) would generate the ratio

$$\frac{\hat{z}_{\text{big}}}{\hat{z}_{\text{small}}} = \left(\frac{p^0_{\text{big}}}{p^0_{\text{small}}} + \frac{p^1_{\text{big}}}{p^1_{\text{small}}}\right)\frac{1}{2} = 5/3 = 1.667\ . \tag{15}$$

Therefore, in formula (10), one could use $\hat{z}_{\text{big}} = 1.667$ and $\hat{z}_{\text{small}} = 1$. Taking the geometric mean (Jevons index) would generate the ratio

$$\frac{\hat{z}_{\text{big}}}{\hat{z}_{\text{small}}} = \sqrt{\frac{p^0_{\text{big}}}{p^0_{\text{small}}}\,\frac{p^1_{\text{big}}}{p^1_{\text{small}}}} = 1.633 \tag{16}$$

and taking the harmonic mean would generate the ratio

$$\frac{\hat{z}_{\text{big}}}{\hat{z}_{\text{small}}} = 2\left(\frac{p^0_{\text{small}}}{p^0_{\text{big}}} + \frac{p^1_{\text{small}}}{p^1_{\text{big}}}\right)^{-1} = 1.6\ .$$

As an alternative to averaging the price ratios (11) and (13), one could also average the prices of one item over time and relate this average value to the corresponding average value of the other item. Using an arithmetic mean would be equivalent to calculating a Dutot index:

$$\frac{\hat{z}_{\text{big}}}{\hat{z}_{\text{small}}} = \frac{p^0_{\text{big}} + p^1_{\text{big}}}{p^0_{\text{small}} + p^1_{\text{small}}} = \frac{8+8}{4+4} = 1.6\ .$$

Taking the harmonic average would also be possible. Obviously, deciding for a geometric average would again generate the result (16).

One should notice that for the value of the generalized unit value index it is irrelevant whether the transformation factors $\hat{z}_{\text{big}} = 1.633$ and $\hat{z}_{\text{small}} = 1$ are used or some uniform proportional transformation of these values. For example, one could use $\hat{z}_{\text{big}} = 8$ and $\hat{z}_{\text{small}} = 4.899$. In this last variant, each item’s transformation factor is computed from the geometric mean of the items’ prices:

$$\hat{z}_{\text{big}} = \sqrt{p_{\text{big}}^0 \cdot p_{\text{big}}^1} = 8 \quad \text{and} \quad \hat{z}_{\text{small}} = \sqrt{p_{\text{small}}^0 \cdot p_{\text{small}}^1} = 4.899 \, .$$

Closer inspection of the price index formula proposed by Davies (1924, p. 185) reveals that it boils down to this last variant.

In this section, seven different representatives of the family of generalized unit value indices have been developed. Each of these members generates its own price index value. It is a common feature of the seven members that the estimation of the transformation factors z_i is exclusively based on the items' prices but not on their quantities. The quantities purchased are relevant in formula (10) but not in the estimation of the transformation factors. It is not difficult to develop a family member that takes account of these quantities. For averaging the base and comparison period prices of an item, one could calculate the item's unit value. In the chocolate example, this approach gives the ratio

$$\frac{\hat{z}_{\text{big}}}{\hat{z}_{\text{small}}} = \frac{\left(p_{\text{big}}^0 x_{\text{big}}^0 + p_{\text{big}}^1 x_{\text{big}}^1 \right) / \left(x_{\text{big}}^0 + x_{\text{big}}^1 \right)}{\left(p_{\text{small}}^0 x_{\text{small}}^0 + p_{\text{small}}^1 x_{\text{small}}^1 \right) / \left(x_{\text{small}}^0 + x_{\text{small}}^1 \right)} = 8/4 = 1.6 \, . \tag{17}$$

Inserting this expression in formula (10) yields the price index formula that has been proposed by Lehr (1885, p. 39). It is probably fair to regard it as the most sophisticated member of the family of generalized unit value indices. For an alternative interpretation of this price index see Balk (2008, p. 8).

6 Comparison to Some Traditional Price Indices

In this study it has been shown that the Paasche index (P_P) and the Laspeyres index (P_L) can be reformulated as generalized unit value indices that estimate the ratios of equivalence from the prices of one period only. In the chocolate example, the index numbers produced by these traditional price indices are

$$P_P = \frac{\sum p_i^1 x_i^1}{\sum p_i^0 x_i^1} = 1.125$$

$$P_L = \frac{\sum p_i^1 x_i^0}{\sum p_i^0 x_i^0} = 1.167 \, .$$

The more sophisticated members of the family of generalized unit value indices represent alternatives to traditional price indices such as the Fisher (P_F), the Marshall-Edgeworth

(P_{ME}), and the Walsh index (P_W). In the chocolate example, the index numbers produced by these traditional price indices are

$$P_F = \sqrt{P_L P_P} = 1.146$$

$$P_{ME} = \frac{\sum p_i^1 (x_i^0 + x_i^1)/2}{\sum p_i^0 (x_i^0 + x_i^1)/2} = 1.143$$

$$P_W = \frac{\sum p_i^1 \sqrt{x_i^0 x_i^1}}{\sum p_i^0 \sqrt{x_i^0 x_i^1}} = 1.145\,.$$

The five listed price index numbers deviate from the result produced by the amended unit value index (P_{AUV} = 1.154) which, knowing the chocolate bars' correct weight ratio, could be viewed as a reference for evaluating other price indices.

Of course, the generalized unit value indices also produce results that deviate from the result of the amended unit value index. The source of the deviation can be seen more clearly, when one expresses the family of generalized unit value indices defined by Equation (10) in the following form:

$$P_{GUV} = \frac{V^1}{V^0} \sum \frac{x_i^1 \hat{z}_i}{\sum x_j^1 \hat{z}_j} \left(\frac{x_i^0}{x_i^1}\right)\,.$$

The first factor on the right hand side of this equation is the value ratio V^1/V^0. The second factor is a weighted average of the quantity ratios (x_i^0/x_i^1). In the chocolate example, the ratio of equivalence $\hat{z}_{big}/\hat{z}_{small}$ is too large when compared with the ratio obtained from the correct but unknown weight ratio. As a consequence, in the weighted average, the large chocolate bars' weight is too large relative to that of small chocolate bars. Since the quantity of large chocolate bars increased over time and the quantity of small chocolate bars remained constant over time, the generalized unit value index produces a smaller value than the amended unit value index. Using the Davies index defined by formula (16), the chocolate example yields

$$P_{GUV} = 1.145.$$

Formula (17), the Lehr index, produces the value

$$P_{GUV} = 1.147.$$

A striking feature of the listed index numbers is the small deviation between the results produced by the sophisticated members of the family of generalized unit value indices (i.e., excluding Laspeyres and Paasche index) and the Fisher, Marshall-Edgeworth, and Walsh index. All these price index formulae produce numbers that take a middle position between the Laspeyres and the Paasche index. Closer inspection reveals that the small deviation does not come as a surprise. If the transformation factors of the generalized unit value index are computed by

$$\hat{z}_i = (p_i^0 + p_i^1)/2\,,$$

this yields the Banerjee index. This index was defined in Equation (2). As pointed out in Section 2, this index is almost identical to the Fisher index defined in Equation (1).

The price indices proposed by Lehr (1885, p. 39) and Davies (1924, p. 185) follow the CPL-approach. Nevertheless, the (by now) traditional price indices (e.g., Laspeyres, Paasche, Fisher, and Walsh index) took centre stage and the proposals of Lehr and Davies were never pursued further. What are the reasons for this neglect? Did price statisticians follow a herd instinct initiated by Irving Fisher's (1922, p. 451) rejection of the CPL-approach? It is possible to prove the reproach of a herd instinct as being invalid, if axiomatic arguments could be advanced against the family of generalized unit value indices. Therefore, the following section investigates the axiomatic properties of the family of generalized unit value indices and compares these properties to those of the most highly regarded traditional price indices.

7 Axiomatic Analysis

In the previous sections, the family of generalized unit value indices P_{GUV} was developed. It was defined in Equation (10):

$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum x_i^0 \hat{z}_i}{\sum x_i^1 \hat{z}_i}\,. \tag{18}$$

For $\hat{z}_i = p_i^1$ one obtains the Laspeyres index and for $\hat{z}_i = p_i^0$ the Paasche index. In both variants the transformation factors are estimated from the prices of only one period. Therefore both index formulae represent two rather crude members of the family of generalized unit value indices. The sophisticated members described in Section 5 utilize the available information of both periods:

$$\hat{z}_i = (p_i^0 x_i^0 + p_i^1 x_i^1)/(x_i^0 + x_i^1) \tag{19}$$

or

$$\hat{z}_i = \sqrt{p_i^0 p_i^1} \tag{20}$$

or

$$\hat{z}_i = (p_i^0 + p_i^1)/2 \tag{21}$$

or $$\hat{z}_i = 2 \cdot [(1/p_i^0) + (1/p_i^1)]^{-1} \tag{22}$$

or $$\hat{z}_i = \left(p_i^0 + \frac{p_1^0}{p_1^1} p_i^1\right) \frac{1}{2} \tag{23}$$

or $$\hat{z}_i = 2 \left(\frac{1}{p_i^0} + \frac{p_1^1}{p_1^0} \frac{1}{p_i^1}\right)^{-1} . \tag{24}$$

As Equation (18) reveals, it would be admissable to multiply all transformation factors z_i with some arbitrary positive constant k. This holds true for all six variants (19) to (24). Multiplying variants (23) and (24) with the constant $1/p_1^0$ yields the formulae

$$\hat{z}_i = \left(\frac{p_i^0}{p_1^0} + \frac{p_i^1}{p_1^1}\right) \frac{1}{2}$$

$$\hat{z}_i = 2 \left(\frac{p_1^0}{p_i^0} + \frac{p_1^1}{p_i^1}\right)^{-1} .$$

From these formulations it can be more easily seen that variants (23) and (24) represent the arithmetic and the harmonic averages of the price ratios.

Are the sophisticated members of the family of generalized unit value indices as attractive as the most highly regarded traditional price indices (e.g., Fisher index)? Axiomatic index theory can contribute to answering this question. It analyses whether a proposed price index formula satisfies a list of postulates (called axioms or tests) that are regarded as indispensable for a meaningful price index formula. However, there is some discussion as to which postulates are convincing axioms and which are not. Therefore, in this study a broad range of axioms is included. The axiomatic properties derived for the family of generalized unit value indices are compared to those of the Fisher, Marshall-Edgeworth, and Walsh index.

It turns out that the sophisticated members of the family of generalized unit value indices defined by Equations (19) to (24) have different axiomatic properties. Table 2 provides an overview of these properties. The postulates (axioms) listed in the first column are formally defined in Appendix A. Proofs of the results of Table 2 are given in Appendix B.

Table 2: Overview of the Axiomatic Properties of the Price Index Formulae (A Filled Triangle Indicates Test Satisfied and an Empty Triangle Indicates Test Violated).

	P_F	P_{ME}	P_W	P_{GUV}							
				(19)	(20)	(21)	(22)	(23)	(24)	P_L	P_P
A1 Strict Mean Value	▲	▲	▲	▽	▲	▲	▲	▲	▲	▲	▲
A2 Proportionality	▲	▲	▲	▽	▲	▲	▲	▲	▲	▲	▲
A3 Identity	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A4 Inv. to Re-Ordering	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A5 Permutation	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽
A6 Inversion	▲	▲	▲	▲	▲	▲	▲	▲	▲	▽	▽
A7 Strict Commens.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A8 Weak Commens.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A9 Price Dimension.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A10 Quant. Dimension.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A11 Strict Quant. Prop.	▲	▽	▲	▽	▲	▲	▲	▲	▲	▲	▲
A12 Weak Quant. Prop.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A13 Lin. Homogeneity	▲	▲	▲	▽	▲	▽	▽	▲	▲	▲	▲
A14 Strict Monotonicity	▲	▲	▲	▽	▽	▲	▽	▽	▽	▲	▲
A15 Weak Monotonicity	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A16 Price Ratio	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A17 Constant Quant.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A18 Time Reversal	▲	▲	▲	▲	▲	▲	▲	▲	▲	▽	▽
A19 Circularity	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽

All listed price indices violate the permutation axiom and the circularity axiom. Almost all sophisticated members of the general unit value index violate the strict monotonicity axiom, the Banerjee index defined by variant (21) being a notable exception. Even though a violation can occur only with extreme intertemporal price and quantity changes, this aspect represents a deficiency of the respective price index formulae. The weak monotonicity axiom is satisfied by all members of the generalized unit value index. The Lehr index defined by variant (19) violates the proportionality axiom and therefore also the strict mean value axiom. Both axioms represent tightenings of the identity axiom. This is true also for the linear homogeneity axiom. This axiom is violated by variants (19), (21), and (22).

The relevance of the axiomatic approach and the individual listed axioms is, and has always been, contested. However, looking at Table 2, it is probably fair to conclude that some members of the family of generalized unit value indices possess an axiomatic profile that is attractive enough to accept that the CPL-approach as their underlying construction principle generates sensible price index formulae.

8 Concluding Remarks

In the CPL-approach, the overall price change is measured as the ratio of the comparison period's average price level to the base period's average price level. Lehr (1885) and Davies (1924) advocated this approach not just for the case of homogeneous or similar items but also for the case of heterogeneous items. For the latter case, Fisher (1911, 1922) regarded the CPL-approach as inadequate and recommended instead the APC-approach. Following his view, the CPL-approach has been sidelined to the case of homogeneous and similar items.

In response, the present study has developed the family of generalized unit value indices. This index family follows the CPL-approach. The study has argued that the family of generalized unit value indices, and therefore, the CPL-approach is also adequate for the case of heterogeneous items. Besides the Banerjee, Lehr, and Davies index, also the Laspeyres and Paasche index are members of this index family. The latter are widely used for the upper level aggregation in official price measurement. In other words, the upper levels of official price measurement rely on index formulae that are consistent with both, the APC-approach and the CPL-approach.

To national statistical institutes, the family of generalized unit value indices can be of additional value. It allows for the price aggregation of similar items with different qualitative characteristics, where hedonic analysis and other sophisticated quality adjustment methods are either not possible or too demanding.

In this study, each member of the family of generalized unit value indices has been examined with respect to its axiomatic properties. Overall, these members exhibit a solid axiomatic record, lending additional support to the claims that in the context of heterogeneous items the CPL-approach is a serious alternative to the APC-approach and that in the context of similar items the family of generalized unit value indices can be of great use.

Besides the axiomatic approach to index theory, the economic and the stochastic approach also exist. In future research one should investigate how the family of generalized unit value indices relates to economic theory. In addition, one could pursue a stochastic approach to evaluating the family of generalized unit value indices. The stochastic approach to index theory usually assumes that all observed price ratios are realisations of the same random variable with an expected value equal to the "common inflation". The CPL-approach suggests pursuing a stochastic analysis that is based on a less controversial assumption: For a given pair of items the observed price ratios of the base period and the comparison period represent realisations of a random variable with an expected value given by the items' ratio of equivalence. Based on this assumption, one could compare the statistical properties of the estimators of the ratios of equivalence used by the various members of the family of generalized unit value indices.

The family of generalized unit value indices can also be applied to other areas of measurement that have not been mentioned in this study. An obvious area is interregional price comparisons. In the context of such comparisons, the transformation factors of the generalized unit value index provide additional flexibility to adjust for regional particularities.

Appendix A

A price index is a function P that maps all prices and quantities of some base period $t = 0$ and comparison period $t = 1$ into a positive index number:

$$P : \mathbb{R}^{4N}_{++} \to \mathbb{R}_{++}, \quad (\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \to P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}),$$

where $\mathbf{p^t} = (p_1^t, \dots, p_N^t)$ and $\mathbf{x^t} = (x_1^t, \dots, x_N^t)$.

A1 The **strict mean value axiom** (Olt, 1996, p. 26) postulates that

$$\min_i\{p_i^1/p_i^0\} < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < \max_i\{p_i^1/p_i^0\},$$

where for $\mathbf{p^1} = \lambda\mathbf{p^0}$ the relation “<” is to be replaced by the relation “=”.

A2 The **proportionality axiom** (Walsh, 1901, p. 115) postulates that

$$P(\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^0}, \mathbf{x^1}) = \lambda, \quad \text{for all } \lambda > 0.$$

A3 The **identity axiom** (Laspeyres, 1871, p. 308) postulates that

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) = 1.$$

A4 The vectors $\tilde{\mathbf{p}}^\mathbf{0}$, $\tilde{\mathbf{x}}^\mathbf{0}$, $\tilde{\mathbf{p}}^\mathbf{1}$, and $\tilde{\mathbf{x}}^\mathbf{1}$ are arbitrary uniform permutations of the vectors $\mathbf{p^0}$, $\mathbf{x^0}$, $\mathbf{p^1}$ and $\mathbf{x^1}$. The **invariance to re-ordering axiom** (Fisher, 1922, p. 63) postulates that

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P(\tilde{\mathbf{p}}^\mathbf{0}, \tilde{\mathbf{x}}^\mathbf{0}, \tilde{\mathbf{p}}^\mathbf{1}, \tilde{\mathbf{x}}^\mathbf{1}).$$

A5 The vectors $\tilde{\mathbf{p}}^\mathbf{0}$ and $\tilde{\mathbf{x}}^\mathbf{0}$ are arbitrary uniform permutations of the vectors $\mathbf{p^0}$ and $\mathbf{x^0}$. The **permutation axiom** (Auer, 2002, p. 534) postulates, that

$$P(\mathbf{p^0}, \mathbf{x^0}, \tilde{\mathbf{p}}^\mathbf{0}, \tilde{\mathbf{x}}^\mathbf{0}) = 1.$$

A6 The vectors $\tilde{\mathbf{p}}^\mathbf{0}$ and $\tilde{\mathbf{x}}^\mathbf{0}$ are special permutations of the vectors $\mathbf{p^0}$ and $\mathbf{x^0}$, such that $p_j^0 = \tilde{p}_k^0, p_k^0 = \tilde{p}_j^0$, $x_j^0 = \tilde{x}_k^0$, $x_k^0 = \tilde{x}_j^0$, and for all items $i \neq j, k$, $p_i^0 = \tilde{p}_i^0$ and $x_i^0 = \tilde{x}_i^0$. The **inversion axiom** (Auer, 2002, p. 534) postulates that

$$P(\mathbf{p^0}, \mathbf{x^0}, \tilde{\mathbf{p}}^\mathbf{0}, \tilde{\mathbf{x}}^\mathbf{0}) = 1.$$

A7 The **strict commensurability axiom** (Pierson, 1896, p. 131) postulates that

$$P(\mathbf{p^0\Lambda}, \mathbf{x^0\Lambda^{-1}}, \mathbf{p^1\Lambda}, \mathbf{x^1\Lambda^{-1}}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}),$$

where Λ is a $N \times N$ diagonal matrix with positive elements λ_i.

A8 The **weak commensurability axiom** (Swamy, 1965, p. 620) postulates that

$$\mathrm{P}\left(\mathbf{p^0}\lambda, \mathbf{x^0}\lambda^{-1}, \mathbf{p^1}\lambda, \mathbf{x^1}\lambda^{-1}\right) = \mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}), \qquad \text{for all } \lambda > 0.$$

A9 The **price dimensionality axiom** (Eichhorn and Voeller, 1976, p. 24) postulates that

$$\mathrm{P}(\lambda\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^1}, \mathbf{x^1}) = \mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}), \qquad \text{for all } \lambda > 0.$$

A10 The **quantity dimensionality axiom** (Funke et al., 1979, p. 680) postulates that

$$\mathrm{P}(\mathbf{p^0}, \lambda\mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^1}) = \mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}), \qquad \text{for all } \lambda > 0.$$

A11 The **strict quantity proportionality axiom** (Vogt, 1980, p. 70, and Diewert, 1992, p. 216) postulates that

$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^1}) = \mathrm{P}(\mathbf{p^0}, \delta\mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = \mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}), \qquad \text{for all } \lambda, \delta > 0.$$

A12 The **weak quantity proportionality axiom** (Auer, 2001, p. 6) postulates that

$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^0}) = \mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^0}), \qquad \text{for all } \lambda > 0.$$

A13 The **linear homogeneity axiom** (Walsh, 1901, p. 385, and Eichhorn and Voeller, 1976, p. 28) postulates that

$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^1}, \mathbf{x^1}) = \lambda\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = \mathrm{P}\left((1/\lambda)\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}\right), \text{ for all } \lambda > 0.$$

A14 Consider two different scenarios for the comparison period ($\mathrm{t} = 1$ and $\mathrm{t} = 1^*$) and the base period ($\mathrm{t} = 0$ and $\mathrm{t} = 0^*$). If for all items $\mathrm{p}_\mathrm{i}^{1^*} \geq \mathrm{p}_\mathrm{i}^1$ and for at least one item i the strict relation holds, then the **strict monotonicity axiom** (Eichhorn and Voeller, 1976, p. 23) postulates that

$$\mathrm{P}\left(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1^*}}, \mathbf{x^1}\right) > P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}),$$

and if for all items $\mathrm{p}_\mathrm{i}^{0^*} \geq \mathrm{p}_\mathrm{i}^0$ and for at least one item i the strict relation holds, then the strict monotonicity axiom postulates that

$$\mathrm{P}\left(\mathbf{p^{0^*}}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}\right) < \mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}).$$

A15 If for all items $\mathrm{p}_\mathrm{i}^1 \geq \mathrm{p}_\mathrm{i}^0$ and for at least one item i the strict relation holds, then the **weak monotonicity axiom** (Olt, 1996, p. 37) postulates that

$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) > P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}).$$

If for all items $\mathrm{p}_\mathrm{i}^1 \leq \mathrm{p}_\mathrm{i}^0$ and for at least one item i the strict relation holds, then the axiom postulates that

$$\mathrm{P}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}) < P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^0},\mathbf{x^1})\,.$$

A16 The **price ratio axiom** (Eichhorn and Voeller, 1990, p. 326) postulates that for $\mathrm{N} = 1$

$$\mathrm{P}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}) = \frac{\mathrm{p}_1^1}{\mathrm{p}_1^0}\,.$$

A17 The **constant quantities axiom** (Lowe, 1822, Appendix, p. 95) postulates that

$$\mathrm{P}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^0}) = \frac{\sum \mathrm{p}_\mathrm{i}^1 \mathrm{x}_1^0}{\sum \mathrm{p}_\mathrm{i}^0 \mathrm{x}_1^0}\,.$$

A18 The **time reversal axiom** (Pierson, 1896, p. 128, and Walsh, 1901, p. 368) postulates that

$$\mathrm{P}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}) = \frac{1}{\mathrm{P}(\mathbf{p^1},\mathbf{x^1},\mathbf{p^0},\mathbf{x^0})}\,.$$

A19 The **circularity axiom** (Westergaard, 1890, p. 218) postulates that

$$\mathrm{P}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^2},\mathbf{x^2}) = \mathrm{P}(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}) \cdot \mathrm{P}(\mathbf{p^1},\mathbf{x^1},\mathbf{p^2},\mathbf{x^2})\,.$$

Appendix B

This appendix sketches out the proofs of the results listed in Table 2. The proofs associated with the Laspeyres ($\mathrm{P_L}$), Paasche ($\mathrm{P_P}$), Fisher ($\mathrm{P_F}$), Marshall-Edgeworth ($\mathrm{P_{ME}}$), and Walsh index ($\mathrm{P_W}$) are either documented in Auer (2001) or they are trivial. The following proofs relate to the sophisticated members of the family of generalized unit value indices ($\mathrm{P_{GUV}}$) defined in Equations (18) to (24). In order to simplify the notation, the hat on the transformation factors is omitted. Furthermore, the term "all sophisticated variants of $\mathrm{P_{GUV}}$" stands for variants (19) to (24).

From Equation (18) it can be directly seen that all sophisticated variants of $\mathrm{P_{GUV}}$ satisfy the **invariance to re-ordering axiom** and the **time reversal axiom**. The **circularity axiom** is violated by all sophisticated variants of $\mathrm{P_{GUV}}$. Only for the special case $\mathrm{z_i} = \mathrm{z}$, it would be satisfied.

In the scenario specified by the **identity axiom** $p_i^0 = p_i^1 = p_i$, all sophisticated variants of P_{GUV} produce $z_i = p_i$ and therefore

$$P_{GUV} = \frac{V^1}{V^0}\frac{\sum p_i\, x_i^0}{\sum p_i\, x_i^1} = \frac{V^1}{V^0}\frac{V^0}{V^1} = 1\,.$$

As a consequence, all these variants satisfy the identity axiom.

In the scenario of the **proportionality axiom** one gets

$$P_{GUV} = \frac{\lambda V^{01}}{V^0}\frac{\sum \tilde{z}_i\, x_i^0}{\sum \tilde{z}_i\, x_i^1},$$

where $\tilde{z}_i$ is the transformation factor associated with the value of λ. The satisfaction of the proportionality axiom requires that

$$\frac{V^{01}}{V^0} = \frac{\sum \tilde{z}_i x_i^1}{\sum \tilde{z}_i x_i^0}$$

and therefore

$$\frac{\sum p_i^0\, x_i^1}{\sum p_i^0\, x_i^0} = \frac{\sum \tilde{z}_i\, x_i^1}{\sum \tilde{z}_i\, x_i^0}\,.$$

This condition is satisfied, if and only if in the scenario specified by the proportionality axiom the ratios of equivalence $\left(\tilde{z}_i/\tilde{z}_j\right)$ are independent from λ and simultaneously coincide with the price ratios of the base period. Except for variant (19), all sophisticated variants of P_{GUV} satisfy these requirements.

If $N = 1$, then (18) yields

$$P_{GUV} = \frac{p^1x^1}{p^0x^0}\frac{x^0z}{x^1z} = \frac{p^1}{p^0}\,.$$

This implies that all sophisticated variants of P_{GUV} satisfy the **price ratio axiom**.

Let $\tilde{z}_i$ indicate the transformation factors resulting from the λ_i-values of the scenario specified by the **strict commensurability axiom**. According to (18), this axiom is satisfied, if and only if

$$\frac{\sum x_i^0\, z_i}{\sum x_i^1 z_i} = \frac{\sum (x_i^0/\lambda_i)\tilde{z}_i}{\sum (x_i^1/\lambda_i)\tilde{z}_i}\,.$$

Variants (19) to (24) give $\tilde{z}_i = z_i\lambda_i$. As a consequence, all sophisticated variants of P_{GUV} satisfy the strict commensurability axiom and therefore also the **weak commensurability axiom.**

The **price dimensionality axiom** is satisfied, if the ratios of equivalence are not affected by a uniform proportional change of all prices. Variants (19) to (24) of P_{GUV} satisfy this requirement. All these variants also satisfy the **quantity dimensionality axiom**, because a price index that satisfies the price dimensionality axiom and the weak commensurability axiom automatically satisfies the quantity dimensionality axiom.

The numerator and the denominator in the second ratio of the right hand side of formula (18) are equal, if and only if

$$\sum z_i(x_1^0 - x_i^1) = 0\,. \tag{25}$$

This is satisfied, if the quantities do not change over time. Therefore, all sophisticated variants satisfy the **constant quantities axiom**.

The scenario specified by the **inversion axiom** yields $V^0 = V^1$. Therefore, the satisfaction of the inversion axiom requires that condition (25) is satisfied. This condition is satisfied, if and only if

$$z_j\left(x_j^0 - x_j^1\right) + z_k(x_k^0 - x_k^1) = 0\,.$$

In the scenario specified by the inversion axiom, this condition is equivalent with the condition

$$z_j\left(x_j^0 - x_k^0\right) + z_k\left(x_k^0 - x_j^0\right) = 0$$

and therefore with the condition

$$\left(z_j - z_k\right)\left(x_j^0 - x_k^0\right) = 0\,.$$

Since $x_j^0 \neq x_k^0$, this condition is satisfied, if and only if

$$z_j = z_k\,.$$

In the scenario specified by the inversion axiom, all sophisticated variants of P_{GUV} satisfy this condition.

The scenario specified by the **permutation axiom** also yields $V^0 = V^1$. However, no variant of P_{GUV} satisfies condition (25).

The **linear homogeneity axiom** is satisfied, if and only if in formula (18) the ratio $[\sum z_i x_i^0]/[\sum z_i x_i^1]$ is not affected by the factor λ, that is, if and only if the ratios of equivalence are independent from the factor λ. Only variants (20), (23), and (24) satisfy this condition.

The **strict quantity proportionality** axiom is satisfied, if and only if

$$\frac{\sum \tilde{z}_i x_i^1}{\sum \tilde{z}_i x_i^0} = \frac{\sum z_i x_i^1}{\sum z_i x_i^0}\,, \tag{26}$$

where $\tilde{z}_i$ indicates the transformation factors associated with the scenario specified by the strict quantity proportionality axiom. Since variants (20) to (24) yield $\tilde{z}_i = z_i$, these variants also satisfy condition (26). Variant (19) does not satisfy this condition. In the scenario specified by the **weak quantity proportionality axiom** one gets $x_i^1 = \lambda x_i^0$. Therefore, expression (18) becomes

$$P_{GUV} = \frac{\lambda V^{10}}{V^0} \frac{\sum z_i x_i^0}{\sum z_i \lambda x_i^0} = \frac{V^{10}}{V^0}\,.$$

Since $V^{10}$and V^0 are independent from λ, all sophisticated variants of P_{GUV} satisfy the weak quantity proportionality axiom.

For the **strict monotonicity** axiom and the **weak monotonicity axiom** one has to consider the case that for all items the relation $dp_k^1 \geq 0$ holds and for at least one item k the strict relation holds. If for this case $dP_{GUV} = \sum(\partial P_{GUV}/\partial p_k^1)dp_k^1 > 0$, then the strict monotonicity axiom is satisfied. From variants (19) to (24) one obtains

(19) $\partial z_k/\partial p_k^1 = x_k^1/(x_k^0 + x_k^1)$

(20) $\partial z_k/\partial p_k^1 = 0{,}5\sqrt{p_k^0/p_k^1}$

(21) $\partial z_k/\partial p_k^1 = 0{,}5$

(22) $\partial z_k/\partial p_k^1 = 2(p_k^1/p_k^0 + 1)^{-2}$

(23) $\partial z_k/\partial p_k^1 = 0{,}5(p_1^0/p_1^1)$

(24) $\partial z_k/\partial p_k^1 = 2(p_1^1/p_1^0)[p_1^1/p_1^0 + p_k^1/p_k^0]^{-2}\,.$

From formula (18) it follows that

$$\begin{aligned}\frac{\partial P_{GUV}}{\partial p_k^1} &= \frac{1}{V^0}\frac{[x_k^1 \sum x_i^0 z_i + V^1 x_k^0(\partial z_k/\partial p_k^1)]\sum x_i^1 z_i - V^1(\sum x_i^0 z_i)x_k^1\,(\partial z_k/\partial p_k^1)}{(\sum x_i^1 z_i)^2}\\ &= \frac{x_k^1 \sum x_i^0 z_i + V^1(\partial z_k/\partial p_k^1)x_k^0 - V^1(\partial z_k/\partial p_k^1)x_k^1\,(\sum x_i^0 z_i)/(\sum x_i^1 z_i)}{V^0 \sum x_i^1\, z_i}\\ &= \frac{x_k^1 \sum x_i^0 z_i[1 - V^1(\partial z_k/\partial p_k^1)/(\sum x_i^1 z_i)] + V^1(\partial z_k/\partial p_k^1)x_k^0}{V^0 \sum x_i^1\, z_i}\,. \qquad (27)\end{aligned}$$

The denominator of (27) is positive. For variant (21), the term in squared brackets simplifies to $[1 - V^1/(V^{01} + V^1)] > 0$. Therefore, variant (21) satisfies the strict monotonicity axiom. However, the other sophisticated variants of P_{GUV} violate this axiom. For sufficiently large values of p_i^1 and x_i^0 $(i \neq k)$, the numerator becomes negative.

In the reference scenario specified by the weak monotonicity axiom $(p_i^0 = p_i^1 = p_i)$ all variants of P_{GUV} give $z_i = p_i$ and $0 < \partial z_k/\partial p_k^1 < 1$. In (27), the term in squared brackets simplifies to $[1 - (\partial z_k/\partial p_k^1)] > 0$. Therefore, the partial derivatives (27), and therefore, also the total differential $\mathrm{dP_{GUV}} = \sum(\partial \mathrm{P_{GUV}}/\partial \mathrm{p}_k^1)\mathrm{dp}_k^1$ are positive. As a consequence, all sophisticated variants of $\mathrm{P_{GUV}}$ satisfy the weak monotonicity axiom.

Since variant (19) violates the proportionality axiom, it also violates the **strict mean value axiom**. A price index that satisfies the linear homogeneity axiom, the weak monotonicity axiom, and the identity axiom, always satisfies the strict mean value axiom. In order to show this, let $\mathrm{p}_i^{1^*} = \mathrm{p}_i^1/\min_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\}$, and therefore,

$$\frac{\mathrm{p}_i^{1^*}}{\mathrm{p}_i^0} = \frac{\mathrm{p}_i^1}{\mathrm{p}_i^0} \cdot \frac{1}{\min_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\}} .$$

As a consequence, one gets $\min_i\{\mathrm{p}_i^{1^*}/\mathrm{p}_i^0\} = 1$. Therefore, for all commodities one obtains $\mathrm{p}_i^{1^*} \geq \mathrm{p}_i^0$. This is a scenario specified by the weak monotonicity axiom. If a price index satisfies this axiom and the identity axiom, then

$$\mathrm{P}\left(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1^*}}, \mathbf{x^1}\right) > 1$$

$$\Rightarrow \qquad \min_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\}\,\mathrm{P}\left(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1^*}}, \mathbf{x^1}\right) > \min_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\} .$$

Due to the satisfaction of the linear homogeneity axiom, this inequality becomes

$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) > \min_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\} .$$

Furthermore, let $\mathrm{p}_i^{1^{**}} = \mathrm{p}_i^1/\max_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\}$, and therefore,

$$\frac{\mathrm{p}_i^{1^{**}}}{\mathrm{p}_i^0} = \frac{\mathrm{p}_i^1}{\mathrm{p}_i^0} \cdot \frac{1}{\max_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\}} .$$

Therefore, for all commodities one obtains $\mathrm{p}_i^{1^{**}} \leq \mathrm{p}_i^0$. This is also a scenario specified by the weak monotonicity axiom. If a price index satisfies this axiom and the identity axiom, then

$$\mathrm{P}\left(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1^{**}}}, \mathbf{x^1}\right) < 1$$

$$\Rightarrow \qquad \max_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\}\,\mathrm{P}\left(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1^{**}}}, \mathbf{x^1}\right) < \max_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\} .$$

Due to the satisfaction of the linear homogeneity axiom, this inequality becomes

$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < \max_j\{\mathrm{p}_j^1/\mathrm{p}_j^0\} .$$

Since variants (20), (23), and (24) satisfy the linear homogeneity axiom, the weak monotonicity axiom, and the identity axiom, they also satisfy the strict mean value axiom.

A price index that satisfies the proportionality axiom and the strict monotonicity axiom also satisfies the strict mean value axiom (earliest proof in Eichhorn and Voeller, 1990, p. 332), because from the proportionality axiom it follows that

$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \min\{\mathrm{p}_i^1/\mathrm{p}_i^0\} \cdot \mathbf{p^0}, \mathbf{x^1}) = \min\{\mathrm{p}_i^1/\mathrm{p}_i^0\}$$

$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \max\{\mathrm{p}_i^1/\mathrm{p}_i^0\} \cdot \mathbf{p^0}, \mathbf{x^1}) = \max\{\mathrm{p}_i^1/\mathrm{p}_i^0\}$$

and from the strict monotonicity axiom it follows that

$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \min\{\mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0\} \cdot \mathbf{p^0}, \mathbf{x^1}) < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})$$
$$\mathrm{P}(\mathbf{p^0}, \mathbf{x^0}, \max\{\mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0\} \cdot \mathbf{p^0}, \mathbf{x^1}) > P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})\,.$$

Taken together, one obtains

$$\min\{\mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0\} < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < \ max\{\mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0\}\,.$$

Therefore, variant (21) also satisfies the strict mean value axiom.

From formula (18), one obtains for variant (22) the expression

$$\mathrm{P}_{\mathrm{GUV}} = \frac{\sum\left(\mathrm{v}_\mathrm{i}^0/\sum \mathrm{v}_\mathrm{j}^0\right)[1 + (\mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0)^{-1}]^{-1}}{\sum\left(\mathrm{v}_\mathrm{i}^1/\sum \mathrm{v}_\mathrm{j}^1\right)[1 + \mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0]^{-1}}\,. \tag{28}$$

If in (28) the price ratios $\mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0$ are replaced by the price ratio $\min_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}$ (the weights $\mathrm{v}_\mathrm{i}^0/\sum \mathrm{v}_\mathrm{j}^0$ remain unchanged), one obtains the expression

$$\frac{\sum\left(\mathrm{v}_\mathrm{i}^0/\sum \mathrm{v}_\mathrm{j}^0\right)\left[1 + \left(\min_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}\right)^{-1}\right]^{-1}}{\sum\left(\mathrm{v}_\mathrm{i}^1/\sum \mathrm{v}_\mathrm{j}^1\right)\left[1 + \min_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}\right]^{-1}} = \min_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}\,.$$

Replacing in the numerator and denominator the price ratio $\min_\mathrm{j}\{\mathrm{p}_\mathrm{j}^0/\mathrm{p}_\mathrm{j}^1\}$ by the actual price ratios $\mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0$, the value of the numerator increases and the value of the denominator falls, leading to

$$\mathrm{P}_{\mathrm{GUV}} > \min_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}\,.$$

If in (28) the price ratios $\mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0$ are replaced by the price ratio $\max_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}$ (the weights $\mathrm{v}_\mathrm{i}^0/\sum \mathrm{v}_\mathrm{j}^o$ remain unchanged), one obtains the expression

$$\frac{\sum\left(\mathrm{v}_\mathrm{i}^0/\sum \mathrm{v}_\mathrm{j}^0\right)\left[1 + \left(\max_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}\right)^{-1}\right]^{-1}}{\sum\left(\mathrm{v}_\mathrm{i}^1/\sum \mathrm{v}_\mathrm{j}^1\right)\left[1 + \max_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}\right]^{-1}} = \max_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}\,.$$

Replacing in the numerator and denominator the price ratio $\max_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}$ by the actual price ratios $\mathrm{p}_\mathrm{i}^1/\mathrm{p}_\mathrm{i}^0$, the value of the numerator falls and the value of the denominator increases, leading to

$$\mathrm{P}_{\mathrm{GUV}} < \max_\mathrm{j}\{\mathrm{p}_\mathrm{j}^1/\mathrm{p}_\mathrm{j}^0\}\,.$$

As a consequence, variant (22) also satisfies the strict mean value axiom.

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